UNIVERSITY OF CAMBRIDGE INTERNATIONAL EXAMINATIONS
General Certificate of Education Advanced Level

FURTHER MATHEMATICS **9231/01**

Paper 1 **October/November 2010**

3 hours

Additional Materials: Answer Booklet/Paper
Graph Paper
List of Formulae (MF10)

5014820134

READ THESE INSTRUCTIONS FIRST

If you have been given an Answer Booklet, follow the instructions on the front cover of the Booklet.
Write your Centre number, candidate number and name on all the work you hand in.
Write in dark blue or black pen.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of a calculator is expected, where appropriate.
Results obtained solely from a graphic calculator, without supporting working or reasoning, will not receive credit.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

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1 The curve C has equation $y = \frac{1}{4}\left(e^{2x} + e^{-2x}\right)$. Show that the length of the arc of C from the point where $x = 0$ to the point where $x = \frac{1}{2}$ is $\dfrac{e^2 - 1}{4e}$. [4]

2 Use the method of differences to find S_N, where

$$S_N = \sum_{n=1}^{N} \frac{1}{n(n+2)}.$$ [4]

Deduce the value of $\lim_{N\to\infty} S_N$. [1]

3 A finite region R in the x-y plane is bounded by the curve with equation $y = \sqrt{x} - \dfrac{1}{\sqrt{x}}$, the x-axis between $x = 1$ and $x = 4$, and the line $x = 4$. Find the exact value of the y-coordinate of the centroid of R. [5]

4 Prove by mathematical induction that, for all non-negative integers n, $7^{2n+1} + 5^{n+3}$ is divisible by 44. [5]

5 Let $I_n = \int_0^1 (1-x)^n \sin x \, dx$ for $n \geqslant 0$. Show that

$$I_{n+2} = 1 - (n+1)(n+2)I_n.$$ [4]

Hence find the value of I_6, correct to 4 decimal places. [4]

6 The linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^4$ is represented by the matrix $\mathbf{A}$, where

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & -1 & \alpha \\ 2 & 3 & -1 & 0 \\ 2 & 1 & 2 & -2 \\ 0 & 1 & -3 & -2 \end{pmatrix}.$$

Given that the dimension of the range space of T is 4, show that $\alpha \neq 1$. [3]

It is now given that $\alpha = 1$. Show that the vectors

$$\begin{pmatrix} 1 \\ 2 \\ 2 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 2 \\ 3 \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 \\ -1 \\ 2 \\ -3 \end{pmatrix}$$

form a basis for the range space of T. [2]

Given also that the vector $\begin{pmatrix} p \\ 1 \\ 1 \\ q \end{pmatrix}$ is in the range space of T, find a condition satisfied by p and q. [3]

7 The roots of the equation $x^3 + 4x - 1 = 0$ are α, β and γ. Use the substitution $y = \dfrac{1}{1+x}$ to show that the equation $6y^3 - 7y^2 + 3y - 1 = 0$ has roots $\dfrac{1}{\alpha+1}$, $\dfrac{1}{\beta+1}$ and $\dfrac{1}{\gamma+1}$. [2]

For the cases $n = 1$ and $n = 2$, find the value of

$$\frac{1}{(\alpha+1)^n} + \frac{1}{(\beta+1)^n} + \frac{1}{(\gamma+1)^n}.$$ [2]

Deduce the value of $\dfrac{1}{(\alpha+1)^3} + \dfrac{1}{(\beta+1)^3} + \dfrac{1}{(\gamma+1)^3}$. [2]

Hence show that $\dfrac{(\beta+1)(\gamma+1)}{(\alpha+1)^2} + \dfrac{(\gamma+1)(\alpha+1)}{(\beta+1)^2} + \dfrac{(\alpha+1)(\beta+1)}{(\gamma+1)^2} = \dfrac{73}{36}$. [3]

8 The curves C_1 and C_2 have polar equations given by

$$C_1 : \quad r = 3\sin\theta, \qquad 0 \leqslant \theta < \pi,$$
$$C_2 : \quad r = 1 + \sin\theta, \quad -\pi < \theta \leqslant \pi.$$

(i) Find the polar coordinates of the points, other than the pole, where C_1 and C_2 meet. [2]

(ii) In a single diagram, draw sketch graphs of C_1 and C_2. [3]

(iii) Show that the area of the region which is inside C_1 but outside C_2 is π. [5]

9 Find the eigenvalues and corresponding eigenvectors of the matrix

$$\mathbf{A} = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{pmatrix}.$$ [7]

Find a non-singular matrix $\mathbf{M}$ and a diagonal matrix $\mathbf{D}$ such that $(\mathbf{A} - 2\mathbf{I})^3 = \mathbf{MDM}^{-1}$, where $\mathbf{I}$ is the 3×3 identity matrix. [3]

10 By using de Moivre's theorem to express $\sin 5\theta$ and $\cos 5\theta$ in terms of $\sin\theta$ and $\cos\theta$, show that

$$\tan 5\theta = \frac{5t - 10t^3 + t^5}{1 - 10t^2 + 5t^4},$$

where $t = \tan\theta$. [5]

Show that the roots of the equation $x^4 - 10x^2 + 5 = 0$ are $\tan\left(\frac{1}{5}n\pi\right)$ for $n = 1, 2, 3, 4$. [2]

By considering the product of the roots of this equation, find the exact value of $\tan\left(\frac{1}{5}\pi\right)\tan\left(\frac{2}{5}\pi\right)$. [3]

11 It is given that $x \neq 0$ and

$$x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 4xy = 8x^2 + 16.$$

Show that if $z = xy$ then

$$\frac{d^2z}{dx^2} + 4z = 8x^2 + 16. \qquad [3]$$

Find y in terms of x, given that $y = 0$ and $\frac{dy}{dx} = -2$ when $x = \frac{1}{2}\pi$. [9]

12 Answer only **one** of the following two alternatives.

EITHER

The curve C has equation

$$y = \frac{x^2 + 2\lambda x}{x^2 - 2x + \lambda},$$

where λ is a constant and $\lambda \neq -1$.

(i) Show that C has at most two stationary points. [3]

(ii) Show that if C has **exactly** two stationary points then $\lambda > -\frac{5}{4}$. [2]

(iii) Find the set of values of λ such that C has two vertical asymptotes. [2]

(iv) Find the x-coordinates of the points of intersection of C with

(a) the x-axis,

(b) the horizontal asymptote.

[3]

(v) Sketch C in each of the cases

(a) $\lambda < -2$,

(b) $\lambda > 2$.

[4]

OR

The plane Π_1 has equation $\mathbf{r} = 2\mathbf{i} + \mathbf{j} + 4\mathbf{k} + \lambda(2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}) + \mu(-\mathbf{i} + \mathbf{k})$. Obtain a cartesian equation of Π_1 in the form $px + qy + rz = d$. [4]

The plane Π_2 has equation $\mathbf{r}.(\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}) = 12$. Find a vector equation of the line of intersection of Π_1 and Π_2. [3]

The line l passes through the point A with position vector $a\mathbf{i} + (2a + 1)\mathbf{j} - 3\mathbf{k}$ and is parallel to $3c\mathbf{i} - 3\mathbf{j} + c\mathbf{k}$, where a and c are positive constants. Given that the perpendicular distance from A to Π_1 is $\frac{15}{\sqrt{6}}$ and that the acute angle between l and Π_1 is $\sin^{-1}\left(\frac{2}{\sqrt{6}}\right)$, find the values of a and c. [7]